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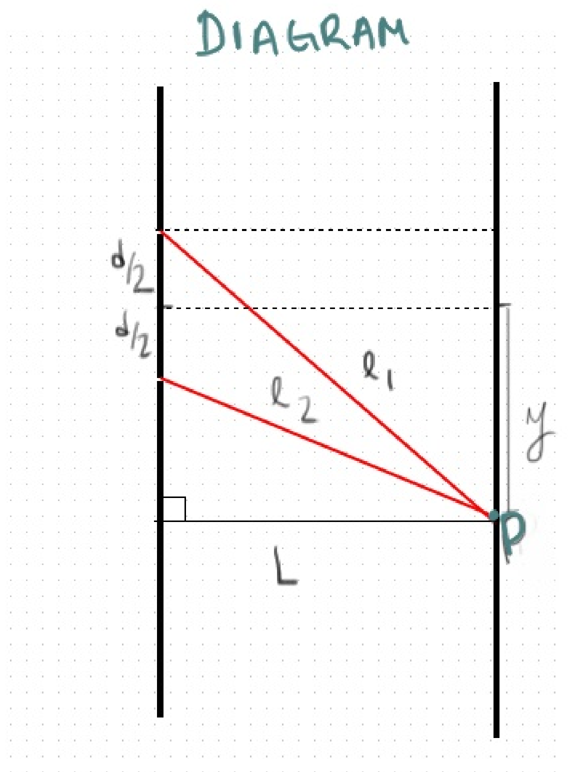
Say you have a double-slit setup, where the slits are of infintesimal width:



To find the intensity $I(y)$ at P, take the equation

$$I(y) = \left| \frac{e^{ikl_1}}{l_1} + \frac{e^{ikl_2}}{l_2} \right|^2 \quad 1$$

where k is the wave number, l_1 is the distance from slit 1 to P, and l_2 is the distance from slit 2 to P.

But that isn't useful because y is not explicitly in the equation. As l_1 and l_2 are related to d, L, and y, one can see the explicit relation of $I(y)$ to y using the Pythagorean Theorem:

$$l_1{}^2 = \left(y + \frac{d}{2}\right)^2 + L^2 \quad ; \quad l_2{}^2 = \left(y - \frac{d}{2}\right)^2 + L^2 \tag{2}$$

$$l_1 = \sqrt{L^2 + \left(y + \frac{d}{2}\right)^2} \quad ; \quad l_2 = \sqrt{L^2 + \left(y - \frac{d}{2}\right)^2} \tag{3}$$

$$l_{1,2} = L\sqrt{1 + \left(\frac{y \pm \frac{d}{2}}{L}\right)^2} \tag{4}$$

Because y and d are assumed to be much smaller than L, the expression $\left(\frac{y \pm \frac{d}{2}}{L}\right)^2$ is very small. This means that the square root can be simplified using a shortened version of binomial expansion. For example, the expression

$$(1 + x^2)^{\frac{1}{2}} \quad \text{can be simplified to} \quad 1 + \frac{x^2}{2} \quad \text{if } x \text{ is very small.}$$

$$l_{1,2} = L\sqrt{1 + \left(\frac{y \pm \frac{d}{2}}{L}\right)^2} \quad \text{can be similarly simplified to} \quad L\left[1 + \frac{1}{2}\left(\frac{y \pm \frac{d}{2}}{L}\right)^2\right] \tag{5}$$

Leaving that for the moment, let us go back to the original equation for $I(y)$ (Eqn. 1) that needed to be simplified.

$$I(y) = \left|\frac{e^{ikl_1}}{l_1} + \frac{e^{ikl_2}}{l_2}\right|^2$$

Remember that y and d were assumed to be much smaller than L. Because of this, Eqn. 5 can be simplified, as L overpowers the equation:

$$l_{1,2} = L\left[1 + \frac{1}{2}\left(\frac{y \pm \frac{d}{2}}{L}\right)^2\right] \simeq L \tag{6}$$

Now, Eqn. 1 can be rewritten as:

$$I(y) = \frac{1}{L^2}\left|e^{ikl_1} + e^{ikl_2}\right|^2 = \frac{1}{L^2}\left(e^{ikl_1} + e^{ikl_2}\right)\left(e^{-ikl_1} + e^{-ikl_2}\right) \tag{7}$$

$$= \frac{1}{L^2}\left(e^0 + e^{ik(l_1-l_2)} + e^{ik(l_2-l_1)} + e^0\right) \quad = \quad \frac{1}{L^2}\left(2 + e^{ik(l_1-l_2)} + e^{ik(l_2-l_1)}\right) \qquad 8$$

The exponent l_1 and l_2 were not simplified thus because an exponential function changes very rapidly and thus even an approximation makes a great deal of change.

Eqn. 8 can be written as

$$\frac{1}{L^2}\left(2 + e^{ik(l_1-l_2)} + e^{-ik(l_1-l_2)}\right) \qquad 9$$

The expression can now takes on the form $e^{ix} + e^{-ix}$, which is equal to $2\cos(x)$!

$$\frac{1}{L^2}\left(2 + e^{ik(l_1-l_2)} + e^{-ik(l_1-l_2)}\right) \quad = \quad \frac{2}{L^2}\left[1 + \cos k\left(l_1 - l_2\right)\right] \qquad 10$$

Using the identity: $2\cos^2 x - 1 = \cos 2x$, it follows that

$$2\cos^2 x = 1 + \cos 2x \quad \text{and thus} \quad 4\cos^2 x = 2\left(1 + \cos^2 x\right)$$

Using the above identity, Eqn. 10 can be rewritten as:

$$\frac{2}{L^2}\left[1 + \cos k\left(l_1 - l_2\right)\right] \quad = \quad \frac{4}{L^2}\cos^2 \frac{k}{2}\left(l_1 - l_2\right) \qquad 11$$

Now, to simplify the argument in the cosine function. Using the expressions for l_1 and l_2 that were derived earlier, this can be done relatively easily.

$$l_{1,2} = L\left[1 + \frac{1}{2}\left(\frac{y \pm \frac{d}{2}}{L}\right)^2\right] \qquad 12$$

$$l_1 - l_2 = \frac{L}{2}\left[\left(\frac{y + \frac{d}{2}}{L}\right)^2 - \left(\frac{y - \frac{d}{2}}{L}\right)^2\right] \quad = \quad \frac{1}{2L}\left[\left(y + \frac{d}{2}\right)^2 - \left(y - \frac{d}{2}\right)^2\right] \qquad 13$$

As this is the difference of two perfect squares, this can be simplified easily:

$$l_1 - l_2 = \frac{1}{2L}\left(y + \frac{d}{2} + y - \frac{d}{2}\right)\left(y + \frac{d}{2} - \left(y - \frac{d}{2}\right)\right) \quad = \quad \frac{yd}{L} \qquad 14$$

This simplification can now be plugged into Eqn. 11.

$$\frac{4}{L^2}\cos^2\frac{k}{2}\left(l_1 - l_2\right) \quad = \quad \frac{4}{L^2}\cos^2\frac{kyd}{2L} \qquad 15$$

$\frac{y}{L} = \tan\theta \simeq \sin\theta$ for very small values of θ. (θ is small because y is very small compared to L). The wavenumber, k can be written as $\frac{2\pi}{\lambda}$. Now Eqn. 15 can also be written as

$$\frac{4}{L^2}\cos^2\frac{kyd}{2L} \quad = \quad \frac{4}{L^2}\cos^2\frac{d\pi\sin\theta}{\lambda} \qquad 16$$

The point of this is to find at what value of θ, or y, will the maxima/ minima of intensity occur. A cosine function reaches its extrema when its argument can be written as $n\pi$, where n is an integer. To find where Eqn. 16 has maxima or minima, its argument must be set equal to $n\pi$. First, this will be done in terms of y.

$$\frac{kyd}{2L} = n\pi \quad ; \quad y = \frac{(2n\pi)\,L}{kd} \qquad 17$$

But remember that $k = \frac{2\pi}{\lambda}$, so

$$y = \frac{nL\lambda}{d} \qquad 18$$

Now the extrema of intensity will be found in terms of θ.

$$\frac{d\pi\sin\theta}{\lambda} = n\pi \quad ; \quad d\sin\theta = n\lambda$$

There's the equation $\lambda = d\sin\theta$!